

# Three Layer $Q_2$ -Free Families in the Boolean Lattice

Jacob Manske\*  
Texas State University  
San Marcos, TX, 78666  
jmanske@txstate.edu

Jian Shen  
Texas State University  
San Marcos, TX 78666  
js48@txstate.edu

August 23, 2011

## Abstract

We prove that the largest  $Q_2$ -free family of subsets of  $[n]$  which contains sets of at most three different sizes has at most  $(3 + 2\sqrt{3})N/3 + o(N) \approx 2.1547N + o(N)$  members, where  $N = \binom{n}{\lfloor n/2 \rfloor}$ . This improves an earlier bound of  $2.207N + o(N)$  by Axenovich, Manske, and Martin.

## 1 Introduction and Motivation

Let  $Q_n$  be the  $n$ -dimensional Boolean lattice corresponding to subsets of an  $n$ -element set ordered by inclusion. A poset  $P = (X, \leq)$  is a subposet of  $Q = (Y, \leq')$  if there is an injective map  $f : X \rightarrow Y$  such that for  $x_1, x_2 \in X$ ,  $x_1 \leq x_2$  implies  $f(x_1) \leq' f(x_2)$ . For a poset  $P$ , we say that a set of elements  $\mathcal{F} \subseteq 2^{[n]}$  is  $P$ -free if  $(\mathcal{F}, \subseteq)$  does not contain  $P$  as a subposet. Let  $ex(n, P)$  be the size of the largest  $P$ -free family of subsets of  $[n]$ . We say that the set of all  $i$ -element subsets of  $[n]$ ,  $\binom{[n]}{i}$ , is the  $i$ th layer of  $Q_n$ . Finally, let  $N(n) = N = \binom{n}{\lfloor n/2 \rfloor}$ ; i.e.,  $N$  is the size of the largest layer of the Boolean lattice.

The first result in this area is Sperner's Theorem [11], which states that  $ex(n, Q_1) = N$ . He also showed that the largest  $Q_1$ -free family is the largest layer in the Boolean lattice.

Many largest  $P$ -free families are simply unions of the largest layers in  $Q_n$ . For instance, the largest  $Q_1$ -free family is simply the largest layer in the Boolean lattice. In [5], Erdős generalized Sperner's result, showing that the size of the family of subsets of  $[n]$  which does not contain a chain with  $k$  elements,  $P_k$ , is equal to the number of elements in the

---

\*Corresponding author.

$k - 1$  largest layers of  $Q_n$ . He also showed that the largest  $P_k$ -free family is the union of the  $(k - 1)$  largest layers in the Boolean lattice.

De Bonis, Katona and Swanepoel show in [4] that  $ex(n, \boxtimes) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$ , where  $\boxtimes$  is a subposet of  $Q_n$  consisting of distinct sets  $a, b, c, d$  such that  $a, b \subset c, d$ . They also showed that if  $n = 3$  or  $n \geq 5$ , the only  $\boxtimes$ -free family which achieves this size is the union of the two largest layers in the Boolean lattice. When  $n = 4$ , there is another construction; take all subsets of size 2 together with  $\{1\}, \{2\}, \{2, 3, 4\}$ , and  $\{1, 3, 4\}$ .

When an exact result is not known, often the asymptotic bounds for  $ex(n, P)$  are expressed in terms of  $N$ . De Bonis and Katona [3] and independently Thanh [12] showed that  $ex(n, V_{r+1}) = N + o(N)$ , where  $V_{r+1}$  is a subposet of  $Q_n$  with distinct elements  $f, g_i$ ,  $i = 1, \dots, r$ ,  $f \subset g_i$  for  $i = 1, \dots, r$ . For a poset  $K_{s,t}$ , with distinct elements  $f_1, \dots, f_s \subset g_1, \dots, g_t$ , and a poset  $P_k(s)$ , with distinct elements  $f_1 \subset \dots \subset f_k \subset g_1, g_2, \dots, g_s$ , Katona and Tarjan [10] and later De Bonis and Katona [3] proved that  $ex(n, K_{s,t}) = 2N + o(N)$  and  $ex(n, P_k(s)) = kN + o(N)$ . Griggs and Katona proved in [6] that  $ex(n, \mathbf{N}) = N + o(N)$ , where  $\mathbf{N}$  is the poset with distinct elements  $a, b, c, d$ , such that  $a \subset c, d$ , and  $b \subset c$ . Griggs and Lu [7] proved that  $ex(n, P_k(s, t)) = (k - 1)N + o(N)$ , where  $P_k(s, t)$  is a poset with distinct elements  $f_1, f_2, \dots, f_s \subset g_2 \subset g_3 \subset \dots \subset g_{k-1} \subset h_1, \dots, h_t$ ,  $k \geq 3$ . They also showed that  $ex(n, O_{4k}) = N + o(N)$ ,  $ex(n, O_{4k-2}) \leq (1 + \sqrt{2}/2)N + o(N)$ , where  $O_i$  is a poset of height two which is a cycle of length  $i$  as an undirected graph. More generally, they proved that if  $G = (V, E)$  is a graph and  $P$  is a poset with elements  $V \cup E$ , with  $v < e$  if  $v \in V$ ,  $e \in E$  and  $v$  incident to  $e$ , then  $ex(n, P) \leq \left(1 + \sqrt{1 - 1/(\chi(G) - 1)}\right)N + o(N)$ , where  $\chi(G)$  is the chromatic number of  $G$ . Bukh [2] proved that  $ex(n, T) = kN + o(N)$ , where  $T$  is a poset whose Hasse diagram is a tree and  $k$  is the integer which is one less than the height of  $T$ . For a more complete survey on the subject, see [9] and [8] for alternate proofs of some of the results listed above.

The smallest poset for which even an asymptotic result is not known is  $Q_2$ . In [1], Axenovich, Martin, and the first author show that  $ex(n, Q_2) \leq 2.283261N + o(N)$  and in the special case where if  $\mathcal{F}$  is a family of subsets of  $[n]$  with at most 3 different sizes and which is  $Q_2$ -free, then  $|\mathcal{F}| \leq 2.207N$ . More recently, Griggs, Li, and Lu were able to show in [9] that  $\lim_{n \rightarrow \infty} \frac{ex(n, Q_2)}{N} \leq 2\frac{3}{11}$  (provided this limit exists), effectively showing that  $ex(n, Q_2) \leq 2\frac{3}{11}N + o(N)$  and thus reducing the leading coefficient in the bound from [1] by about .0105. Our main result focuses on the special case where  $\mathcal{F}$  contains sets of at most 3 sizes; we state the result below as Theorem 1.

**Theorem 1** *Let  $n$  be a positive integer. If  $\mathcal{F} \subset Q_n$  is a  $Q_2$ -free family,  $\mathcal{F} = \mathcal{S} \cup \mathcal{T} \cup \mathcal{U}$ , where  $\mathcal{S}$  is a collection of minimal elements of  $\mathcal{F}$ ,  $\mathcal{U}$  is a collection of maximal elements of  $\mathcal{F}$  and  $\mathcal{T} = \mathcal{F} \setminus (\mathcal{S} \cup \mathcal{U})$  such that for any  $T \in \mathcal{T}$ ,  $S \in \mathcal{S}$ ,  $U \in \mathcal{U}$ ,  $|T| = k$ ,  $|U| > k$ ,  $|S| < k$ , then  $|\mathcal{F}| \leq (3 + 2\sqrt{3})N/3 + o(N) \approx 2.1547N + o(N)$ . In particular, if  $\mathcal{F}$  is a*

$Q_2$ -free subset of three layers of  $Q_n$ , then  $|\mathcal{F}| \leq 2.1547N + o(N)$ .

## 2 Proof of Theorem 1

Following the argument in [1], it suffices to prove Theorem 1 in the case where  $\mathcal{F}$  contains sets of size  $k$ ,  $(k-1)$ , and  $(k+1)$ .

For two functions  $A(n)$  and  $B(n)$ , by  $A(n) \lesssim B(n)$  (or  $B(n) \gtrsim A(n)$ ) we mean

$$\text{either } A(n) \leq B(n) \text{ or } \lim_{n \rightarrow \infty} \frac{A(n)}{B(n)} = 1.$$

Suppose  $\mathcal{F}$  is a  $Q_2$ -free family from three layers,  $L_1, L_2, L_3$ , of the Boolean lattice  $Q_n$ , where  $L_1 = \binom{[n]}{k-1}$ ,  $L_2 = \binom{[n]}{k}$ ,  $L_3 = \binom{[n]}{k+1}$ , and by Lemma 1 from [1], we may assume  $n/2 - n^{2/3} \leq k \leq n/2 + n^{2/3}$ . Let  $\mathcal{S} = \mathcal{F} \cap L_1$ ,  $\mathcal{T} = \mathcal{F} \cap L_2$ ,  $\mathcal{U} = \mathcal{F} \cap L_3$ .

For  $X \in L_1$ ,  $Y \in L_2$ , and  $Z \in L_3$ , we define

$$\begin{aligned} f(X) &= |\{T \in \mathcal{T} : X \subset T\}|; & g(Z) &= |\{T \in \mathcal{T} : Z \supset T\}|; \\ \check{f}(Y) &= |\{S \in \mathcal{S} : S \subset Y\}|; & \check{g}(Y) &= |\{U \in \mathcal{U} : U \supset Y\}|; \end{aligned}$$

Note that

$$\sum_{X \in \mathcal{S}} f(X) = \sum_{Y \in \mathcal{T}} \check{f}(Y) \text{ and } \sum_{Z \in \mathcal{U}} g(Z) = \sum_{Y \in \mathcal{T}} \check{g}(Y).$$

From [1], we have

$$|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}| \leq 2N - |\mathcal{T}| + \frac{1}{k} \sum_{Y \in \mathcal{T}} (\check{f}(Y) + \check{g}(Y)), \quad (1)$$

where  $N = \binom{n}{\lfloor n/2 \rfloor} \approx \binom{n}{k}$ . We start with a few lemmas involving some counting arguments.

**Lemma 1** *For any  $X \in \mathcal{S}$  and  $Y \in \mathcal{T}$  with  $X \subset Y$ ,*

$$f(X) + \check{g}(Y) \leq n - k + 1 \lesssim k.$$

*For any  $Y \in \mathcal{T}$  and  $Z \in \mathcal{U}$  with  $Y \subset Z$ ,*

$$g(Z) + \check{f}(Y) \leq k + 1 \lesssim k.$$

**Proof.** We only prove  $f(X) + \check{g}(Y) \leq n - k + 1 \lesssim k$ , and the other inequality follows similarly. By definition,  $\check{g}(Y) = |\{U \in \mathcal{U} : U \supset Y\}| \leq n - |Y| = n - k$ . So we may suppose without loss of generality that  $f(X) \geq 2$ . For any  $Y' \in \mathcal{T}$  with  $X \subset Y' \neq Y$ , we have  $|Y \cup Y'| = |Y| + |Y'| - |Y \cap Y'| = |Y| + |Y'| - |X| = 2k - (k-1) = k+1$  and thus

$Y \cup Y' \in L_3$ . Since  $\mathcal{F}$  is  $Q_2$ -free, we have  $Y \cup Y' \in (L_3 - \mathcal{U})$ . Further  $Y \cup Y' \neq Y \cup Y''$  for any other  $Y''$  with  $X \subset Y'' \in (\mathcal{T} - \{Y, Y'\})$ . Therefore,

$$\begin{aligned} \check{g}(Y) &\leq |\{U \in \mathcal{U} : U \supset Y \text{ and } U \neq Y \cup Y' \text{ with } X \subset Y' \in (\mathcal{T} - \{Y\})\}| \\ &= |\{U \in \mathcal{U} : U \supset Y\}| - |\{Y' \in \mathcal{T} : X \subset Y' \in (\mathcal{T} - \{Y\})\}| \\ &= n - k - f(X) + 1. \end{aligned}$$

□

**Lemma 2**

$$\begin{aligned} |\mathcal{S}| &\gtrsim \sum_{Y \in \mathcal{T}} \frac{\check{f}(Y)}{k - \check{g}(Y)}, \\ |\mathcal{U}| &\gtrsim \sum_{Y \in \mathcal{T}} \frac{\check{g}(Y)}{k - \check{f}(Y)}. \end{aligned}$$

**Proof.** We use double counting to only prove the first inequality, and the other inequality follows symmetrically. First

$$\sum_{(X,Y): \mathcal{S} \ni X \subset Y \in \mathcal{T}} \frac{1}{f(X)} = \sum_{X \in \mathcal{S}} \sum_{X \subset Y \in \mathcal{T}} \frac{1}{f(X)} = \sum_{X \in \mathcal{S}} 1 = |\mathcal{S}|.$$

Second, by Lemma 1,

$$\sum_{(X,Y): \mathcal{S} \ni X \subset Y \in \mathcal{T}} \frac{1}{f(X)} = \sum_{Y \in \mathcal{T}} \sum_{\mathcal{S} \ni X \subset Y} \frac{1}{f(X)} \gtrsim \sum_{Y \in \mathcal{T}} \sum_{\mathcal{S} \ni X \subset Y} \frac{1}{k - \check{g}(Y)} = \sum_{Y \in \mathcal{T}} \frac{\check{f}(Y)}{k - \check{g}(Y)}.$$

□

**Lemma 3** For any non-negative reals  $x$  and  $y$  with  $x < k$ ,  $y < k$ , and  $x + y \geq k$ ,

$$\frac{x}{k - y} + \frac{y}{k - x} \geq \frac{2x + 2y}{2k - x - y}.$$

**Proof.**

$$x(k - x)(2k - x - y) + y(k - y)(2k - x - y) - (2x + 2y)(k - x)(k - y) = (x - y)^2(x + y - k) \geq 0.$$

□

Define  $\mathcal{T}_1 := \{Y \in \mathcal{T} : \check{f}(Y) + \check{g}(Y) \geq k\}$  and  $\mathcal{T}_2 := \{Y \in \mathcal{T} : \check{f}(Y) + \check{g}(Y) < k\}$ .

**Lemma 4**

$$\sum_{Y \in \mathcal{T}_2} \left( \check{f}(Y) + \check{g}(Y) \right) \leq k|\mathcal{T}_2|,$$

$$\sum_{Y \in \mathcal{T}_1} \left( \check{f}(Y) + \check{g}(Y) \right) \lesssim \frac{2k|\mathcal{T}_1|(|\mathcal{S}| + |\mathcal{U}|)}{|\mathcal{S}| + |\mathcal{U}| + 2|\mathcal{T}_1|}.$$

**Proof.** By the definition of  $\mathcal{T}_2$ , we have  $\sum_{Y \in \mathcal{T}_2} \left( \check{f}(Y) + \check{g}(Y) \right) \leq \sum_{Y \in \mathcal{T}_2} k = k|\mathcal{T}_2|$ . We now prove the second inequality of the lemma. Recall that  $\mathcal{T}_1 \cup \mathcal{T}_2$  forms a disjoint union of  $\mathcal{T}$ . By Lemma 2 and Lemma 3 (with  $x = \check{f}(Y)$  and  $y = \check{g}(Y)$ ),

$$\begin{aligned} |\mathcal{S}| + |\mathcal{U}| &\gtrsim \sum_{Y \in \mathcal{T}} \left( \frac{\check{f}(Y)}{k - \check{g}(Y)} + \frac{\check{g}(Y)}{k - \check{f}(Y)} \right) \\ &\geq \sum_{Y \in \mathcal{T}_1} \frac{2\check{f}(Y) + 2\check{g}(Y)}{2k - \check{f}(Y) - \check{g}(Y)} \\ &= \sum_{Y \in \mathcal{T}_1} \left( \frac{4k}{2k - \check{f}(Y) - \check{g}(Y)} - 2 \right) \\ &= \sum_{Y \in \mathcal{T}_1} \frac{4k}{2k - \check{f}(Y) - \check{g}(Y)} - 2|\mathcal{T}_1|. \end{aligned}$$

Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &(|\mathcal{S}| + |\mathcal{U}| + 2|\mathcal{T}_1|) \left( 2k|\mathcal{T}_1| - \sum_{Y \in \mathcal{T}_1} \left( \check{f}(Y) + \check{g}(Y) \right) \right) \\ &\gtrsim \sum_{Y \in \mathcal{T}_1} \frac{4k}{2k - \check{f}(Y) - \check{g}(Y)} \sum_{Y \in \mathcal{T}_1} \left( 2k - \check{f}(Y) - \check{g}(Y) \right) \geq 4k|\mathcal{T}_1|^2, \end{aligned}$$

which is equivalent to

$$\sum_{Y \in \mathcal{T}_1} \left( \check{f}(Y) + \check{g}(Y) \right) \lesssim 2k|\mathcal{T}_1| - \frac{4k|\mathcal{T}_1|^2}{|\mathcal{S}| + |\mathcal{U}| + 2|\mathcal{T}_1|} = \frac{2k|\mathcal{T}_1|(|\mathcal{S}| + |\mathcal{U}|)}{|\mathcal{S}| + |\mathcal{U}| + 2|\mathcal{T}_1|}.$$

□

We are now ready to prove Theorem 1.

**Proof.** We will show that

$$|\mathcal{S}| + |\mathcal{U}| + |\mathcal{T}| \lesssim \frac{3 + 2\sqrt{3}}{3} N \approx 2.1547N.$$

By (1) and Lemma 4,

$$\begin{aligned}
|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}| &\lesssim 2N - |\mathcal{T}| + \frac{1}{k} \sum_{Y \in \mathcal{T}} \left( \check{f}(Y) + \check{g}(Y) \right) \\
&= 2N - |\mathcal{T}_1| - |\mathcal{T}_2| + \frac{1}{k} \sum_{Y \in \mathcal{T}_1} \left( \check{f}(Y) + \check{g}(Y) \right) + \frac{1}{k} \sum_{Y \in \mathcal{T}_2} \left( \check{f}(Y) + \check{g}(Y) \right) \\
&\lesssim 2N - |\mathcal{T}_1| + \frac{2|\mathcal{T}_1| (|\mathcal{S}| + |\mathcal{U}|)}{|\mathcal{S}| + |\mathcal{U}| + 2|\mathcal{T}_1|} \\
&= 2N + \frac{|\mathcal{T}_1| (|\mathcal{S}| + |\mathcal{U}| - 2|\mathcal{T}_1|)}{|\mathcal{S}| + |\mathcal{U}| + 2|\mathcal{T}_1|} \\
&\lesssim 2N + (|\mathcal{U}| + |\mathcal{T}_1| + |\mathcal{S}|) \cdot f(x),
\end{aligned}$$

where

$$f(x) = \frac{x - 2}{(x + 2)(x + 1)}$$

with  $x = \frac{|\mathcal{S}| + |\mathcal{U}|}{|\mathcal{T}_1|} > 0$ . Note that the function  $f(x)$  has a unique critical point at  $x = (2 + 2\sqrt{3})$  on the interval  $[0, \infty)$ . The function  $f(x)$  achieves its maximum value at  $x = (2 + 2\sqrt{3})$  and thus

$$|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}| \lesssim 2N + (|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}|) \cdot f(2 + 2\sqrt{3}) = 2N + (7 - 4\sqrt{3}) (|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}|),$$

from which we have  $|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}| \lesssim (3 + 2\sqrt{3}) N/3 \approx 2.1547N$ .  $\square$

### 3 Future work

There are two ways of extending the argument to get a general bound on  $ex(n, Q_2)$ . One is to adapt the counting argument above to work with a family of subsets of  $[n]$  with more than 3 sizes. Another way is to show that if  $\mathcal{F}$  is a family of size  $ex(n, Q_2)$ , then  $\mathcal{F}$  contains sets of at most 3 different sizes. If the latter is true, then Theorem 1 shows that  $ex(n, Q_2) \leq 2.1547N$ .

We may also investigate  $\pi(Q_2) = \lim_{n \rightarrow \infty} \frac{ex(n, Q_2)}{N}$ , as the authors in [9] do. It is not known if  $\pi(Q_2)$  exists, although it is conjectured in [7] that  $\pi(P)$  exists and is an integer for any finite poset  $P$ . If true, then  $\pi(Q_2) = 2$  and  $ex(n, Q_2) = 2N + o(N)$ .

**Acknowledgments.** Shen's research was partially supported by NSF (CNS 0835834, DMS 1005206) and Texas Higher Education Coordinating Board (ARP 003615-0039-2007).

## References

- [1] M. Axenovich, J. Manske, and R. Martin.  $Q_2$ -free families of the Boolean lattice. *accepted by Order*, 2011.
- [2] B. Bukh. Set families with a forbidden subposet. *Electron. J. Combin.*, 16(1):Research paper 142, 11, 2009.
- [3] A. De Bonis and G. O. H. Katona. Largest families without an  $r$ -fork. *Order*, 24(3):181–191, 2007.
- [4] A. De Bonis, G. O. H. Katona, and K. J. Swanepoel. Largest family without  $A \cup B \subseteq C \cap D$ . *J. Combin. Theory Ser. A*, 111(2):331–336, 2005.
- [5] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.
- [6] J. R. Griggs and G. O. H. Katona. No four subsets forming an  $N$ . *J. Combin. Theory Ser. A*, 115(4):677–685, 2008.
- [7] J. R. Griggs and L. Lu. On families of subsets with a forbidden subposet. *Combinatorics, Probability and Computing*, 18:731–748, 2009.
- [8] J.R. Griggs and W.-T. Li. The partition method for poset-free families. *preprint*, 2011.
- [9] J.R. Griggs, W.-T. Li, and L. Lu. Diamond-free families. *accepted by J. Combinatorial Theory (Ser. A)*, 2011.
- [10] G. O. H. Katona and T. G. Tarján. Extremal problems with excluded subgraphs in the  $n$ -cube. In *Graph theory (Lagów, 1981)*, volume 1018 of *Lecture Notes in Math.*, pages 84–93. Springer, Berlin, 1983.
- [11] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27(1):544–548, 1928.
- [12] H. T. Thanh. An extremal problem with excluded subposet in the Boolean lattice. *Order*, 15(1):51–57, 1998.